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Relations between Rational and Spline Approximations in L_p Metric

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1. INTRODUCTION

In this paper we show that the rational functions are not worse than the spline functions as a tool for approximation in L_p metric $(1 \leqslant p < \infty)$.

Denote by H_n the set of all algebraic polynomials of nth degree with real coefficients and by R_n the set of all rational functions of nth degree, i.e., $R \in R_n$ if $R = P_1/P_2, P_1, P_2 \in H_n$. By $S(k, n, [a, b])$ we shall denote the set of all piecewise polynomial functions (spline functions with defect) of degree $k-1$ with $n+1$ free knots on $[a, b]$, i.e., $\varphi \in S(k, n, [a, b])$ if there exists $n+1$ points $x_0 = a < x_1 < \cdots < x_n = b$ such that in each interval (x_{i-1}, x_i), φ is an algebraic polynomial of degree $k-1$.

Denote by $R_n(f)_p = R_n(f, [a, b])_p$ and $S_n^k(f)_p = S_n^k(f, [a, b])_p$ the best approximations of $f \in L_p[a, b]$, $1 \leqslant p \leqslant \infty$, by means of the elements of R_n and $S(k, n, [a, b])$, respectively, in L_p metric, i.e.,

$$R_n(f)_p = R_n(f, [a, b])_p = \inf\{\|f - R\|_p : R \in R_n\},$$
$$S_n^k(f) = S_n^k(f, [a, b])_p = \inf\{\|f - \varphi\|_p : \varphi \in S(k, n, [a, b])\}.$$

By $\|f\|_p = \|f\|_{L_p[a,b]}$ we shall denote the L_p norm of f on $[a, b]$ $(1 \leqslant p \leqslant \infty)$.

Throughout the paper $C, C_1, C_2, \ldots, D, D_1, \ldots$ denote positive constants depending only on the corresponding parameters which can be written in brackets.

It is well known that the rational and spline approximations of functions

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are closely connected. V. A. Popov [1] proved the following estimate: If $f \in C_{[0,1]}$, then for $n \geqslant 1$

$$S_n^1(f)_\infty \leqslant 2^7 n^{-1} \sum_{\nu=0}^{n} R_\nu(f)_\infty .$$

In [2] we have obtained the following two estimates:

(i) If $f \in L_{p+\varepsilon}[a, b]$, $1 \leqslant p < \infty$, $\varepsilon > 0$, $k \geqslant 1$ and $\alpha > 0$, then for $n \geqslant \max\{1, k-1\}$

$$R_n(f)_p \leqslant Cn^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1} S_\nu^k(f)_{p+\varepsilon}, \tag{1}$$

where $C = C(p, k, \varepsilon, \alpha, a, b)$.

(ii) If $f \in L_{p+\varepsilon}[a, b]$, $1 \leqslant p < \infty$, $\varepsilon > 0$, $k \geqslant 1$ and $0 < \alpha < k$, then for $n \geqslant 1$

$$S_n^k(f)_p \leqslant Cn^{-\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f)_{p+\varepsilon}.$$

V. A. Popov [3] obtained connection between the rational uniform approximation of functions and the polynomial L_p $(p > 1)$ approximation of their derivatives. In [4, 5] we have proved some connections between the rational and spline approximations of functions and their derivatives in different L_p metrics. Yu. A. Brudnyi [6] announced some results, which are closely connected to ours.

The main result of this paper is announced in [7].

2. The Main Result

The aim of this paper is to prove the following improvement of the estimate (1).

THEOREM 2.1. *If $f \in L_p[a, b]$, $1 \leqslant p < \infty$, $k \geqslant 1$ and $\alpha > 0$, then for $n \geqslant \max\{1, k-1\}$*

$$R_n(f)_p \leqslant Cn^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1} S_\nu^k(f)_p, \tag{2}$$

where $C = C(p, k, \alpha)$.

Moreover, if we put $f(x)=0$ *for* $x\in(-\infty,\infty)\setminus[a,b]$, *then for* $n\geqslant k-1$

$$R_n(f,(-\infty,\infty))_p\leqslant Cn^{-\alpha}\left\{\|f\|_p+\sum_{\nu=1}^{n}\nu^{\alpha-1}S_\nu^k(f,[a,b])_p\right\},\qquad(3)$$

where $C=C(p,k,\alpha)$.

Clearly the estimates (2) and (3) do not hold true for $p=\infty$.

COROLLARY 2.1. *If* $S_n^k(f)_p=O(n^{-\gamma})$, $1\leqslant p<\infty$, $\gamma>0$, *then* $R_n(f)_p=O(n^{-\gamma})$.

Theorem 2.1 can be used successfully in more general situations but for rates of convergence not better than $O(n^{-\gamma})$.

COROLLARY 2.2. *If* $S_n^k(f)_p=O(n^{-\gamma}\omega(n^{-1}))$, *where* $1\leqslant p<\infty$, $k\geqslant1$, $\gamma\geqslant0$, *and* ω *is any increasing on* $[0,\infty)$ *function which tends to zero as* $\delta\to0+$ *and* $\omega(2\delta)\leqslant2\omega(\delta)$ *for* $\delta\geqslant0$ *then* $R_n(f)_p=O(n^{-\gamma}\omega(n^{-1}))$.

3. AUXILIARY RESULTS

The proof of Theorem 2.1 is based on the following statement:

THEOREM 3.1. *Let* $\varphi\in S(k,m,[a,b])$, *where* $k\geqslant1$, $m\geqslant1$ *and* $[a,b]$ *is an arbitrary compact interval, and let* $1\leqslant p<\infty$. *Put* $\varphi(x)=0$ *for* $x\in(-\infty,\infty)\setminus[a,b]$. *Then for each* $\lambda>0$ *there is a rational function* R *such that*

$$\deg R\leqslant Dm\ln^2\left(e+\frac{1}{\lambda}\right)$$

and

$$\|\varphi-R\|_{L_p(-\infty,\infty)}\leqslant\lambda\|\varphi\|_{L_p[a,b]},$$

where $D=D(p,k)>1$.

The same statement in another form: In the above assumptions the following estimate holds true for $n\geqslant1$:

$$R_n(\varphi,(-\infty,\infty))_p\leqslant2\exp\left\{-C\sqrt{\frac{n}{m}}\right\}\|\varphi\|_{L_p[a,b]},$$

where $C=C(p,k)>0$.

In order to prove Theorem 3.1 we need the following lemma.

LEMMA 3.1. *Let $Q \in H_k$, $k \geqslant 0$, $1 \leqslant p < \infty$, and let $\Delta = [a, b]$ be an arbitrary compact interval. Then for each $\lambda > 0$ there is a rational function R such that*

$$\deg R \leqslant D \ln^2\left(e + \frac{1}{\lambda}\right),$$

$$\|Q - R\|_{L_p(\Delta)} \leqslant D\lambda \|Q\|_{L_p(\Delta)}$$

and

$$|R(x)| \leqslant D\left(\frac{|\Delta|}{|\Delta| + \rho(x, \Delta)}\right)^4 \frac{\lambda \|Q\|_{L_p(\Delta)}}{|\Delta|^{1/p}} \qquad \text{for} \quad x \in (-\infty, \infty)\backslash\Delta,$$

where

$$D = D(p, k), \qquad \rho(x, \Delta) = \begin{cases} a - x & \text{for} \quad x \leqslant a \\ 0 & \text{for} \quad x \in \Delta \\ x - b & \text{for} \quad x \geqslant b \end{cases}$$

is the distance from the point x to the interval Δ and $|\Delta| = b - a$.

The proof of Lemma 3.1 is based on the following two lemmas:

LEMMA 3.2. *If $Q \in H_k$, $k \geqslant 0$, $1 \leqslant p < \infty$ and $\Delta = [-b, b]$, $b > 0$, then for $x \in (-\infty, \infty)\backslash\Delta$*

$$|Q(x)| \leqslant C \frac{\|Q\|_{L_p(\Delta)} |x|^k}{|\Delta|^{k + 1/p}},$$

where $C = C(k)$.

Proof. Lemma 3.2 follows immediately from the following well-known inequalities:

(i) If $Q \in H_k$, then for $|x| \geqslant 1$

$$\begin{aligned} |Q(x)| &\leqslant \|Q\|_{L_\infty[-1,1]} |T_k(x)| \leqslant \|Q\|_{L_\infty[-1,1]} (|x| + \sqrt{x^2 - 1})^k \\ &\leqslant 2^k \|Q\|_{L_\infty[-1,1]} |x|^k, \end{aligned}$$

where $T_k(x) = \cos(k \arccos x)$ for $|x| \leqslant 1$; see [8]. Consequently, if $Q \in H_k$, then for $|x| \geqslant b$

$$|Q(x)| \leqslant 2^k \|Q\|_{L_\infty[-b,b]} \frac{|x|^k}{b^k} = 2^{2k} \frac{\|Q\|_{L_\infty(\Delta)} |x|^k}{|\Delta|^k}.$$

(ii) If $Q \in H_k$, then

$$\|Q\|_{L_\infty(\Delta)} \leqslant C(k) \frac{\|Q\|_{L_p(\Delta)}}{|\Delta|^{1/p}}. \quad \blacksquare$$

LEMMA 3.3. *If $\alpha > 0$, $0 < \delta \leqslant 1$, $0 < \gamma \leqslant 1$ and $r \geqslant 0$, r – integer, then there is a rational function σ such that*

$$\deg \sigma \leqslant B \ln\left(e + \frac{1}{\delta}\right) \ln\left(e + \frac{1}{\gamma}\right) + 4r, \tag{4}$$

$$0 \leqslant 1 - \sigma(x) \leqslant \gamma \qquad \textit{for} \quad |x| \leqslant d - \delta d, \tag{5}$$

$$0 \leqslant \sigma(x) \leqslant \left(\frac{2d}{d + |x|}\right)^{4r} \gamma \qquad \textit{for} \quad |x| \geqslant d + \delta d \tag{6}$$

and

$$0 \leqslant \sigma(x) \leqslant 1 \qquad \textit{for} \quad x \in (-\infty, \infty), \tag{7}$$

where $B > 1$ is an absolute constant.

Proof. By Lemma 2 in [9] it follows that if $0 \leqslant \varepsilon < \frac{1}{2}$ and $n \geqslant 1$ then the rational function $S(x) = P(x)/P(-x)$, $P(x) = \prod_{i=1}^{n} (x - \varepsilon^{i/n})$, $S \in R_n$, satisfies

$$|S(x)| \geqslant \frac{1}{C_1} \exp\left(\frac{C_2 n}{\ln(1/\varepsilon)}\right) \qquad \text{for} \quad x \in [-1, -\varepsilon] \tag{8}$$

and

$$|S(x)| \leqslant C_1 \exp\left(-\frac{C_2 n}{\ln(1/\varepsilon)}\right) \qquad \text{for} \quad x \in [\varepsilon, 1], \tag{9}$$

where $C_1, C_2 > 0$ are absolute constants.

Using the notations above, put

$$\varepsilon = \frac{1}{e + 2/\delta}, \qquad n = \left[\frac{1 + C_1^2}{2C_2} \ln \frac{1}{\varepsilon} \ln\left(e + \frac{1}{\gamma}\right) + 1\right],$$

$$\sigma_1(x) = \frac{1}{S^2(x)((1-x)/(1+x))^{2r} + 1},$$

where $[x]$ denotes the integer part of x.

Obviously

$$\deg \sigma_1 = 2n + 2r \leqslant B_1 \ln\left(e + \frac{1}{\delta}\right) \ln\left(e + \frac{1}{\gamma}\right) + 2r, \tag{10}$$

where $B_1>0$ is an absolute constant. By (8) and the choice of ε and n we get for $x\in[-1,-\varepsilon]\supset[-1,-\delta/2]$

$$|\sigma_1(x)|\leqslant\frac{(1+x)^{2r}}{S^2(x)}\leqslant\frac{C_1^2(1+x)^{2r}}{\exp\left\{\dfrac{2C_2n}{\ln(1/\varepsilon)}\right\}}\leqslant(1+x)^{2r}\gamma \tag{11}$$

and by (9) we get for $x\in[\varepsilon,1]\supset[\delta/2,1]$

$$|1-\sigma_1(x)|=\frac{S^2(x)\left(\dfrac{1-x}{1+x}\right)^{2r}}{S^2(x)\left(\dfrac{1-x}{1+x}\right)^{2r}+1}\leqslant(1-x)^{2r}S^2(x)\leqslant(1-x)^{2r}\gamma \tag{12}$$

Clearly

$$0\leqslant\sigma_1(x)\leqslant 1 \qquad \text{for} \quad x\in(-\infty,\infty). \tag{13}$$

Consider the rational function

$$\sigma(x)=\sigma_1(\varphi(x)), \qquad \varphi(x)=\frac{d^2-x^2}{d^2+x^2}.$$

We shall show that σ satisfies (4)–(7). Indeed, by (10) we obtain

$$\begin{aligned}\deg\sigma=2\deg\sigma_1&\leqslant 2B_1\ln\left(e+\frac{1}{\delta}\right)\ln\left(e+\frac{1}{\gamma}\right)+4r\\&=B\ln\left(e+\frac{1}{\delta}\right)\ln\left(e+\frac{1}{\varphi}\right)+4r,\end{aligned}$$

i.e., σ satisfies (4). Obviously, (13) implies (7). It remains to show that σ satisfies (5) and (6). Clearly, the function φ is even, strictly decreasing on $[0,\infty)$, $\varphi(0)=1$, $\varphi(d)=0$, $\lim_{x\to+\infty}\varphi(x)=-1$. Since $\varphi(d-\delta d)\geqslant\delta/2$ and $\varphi(d+\delta d)\leqslant-\delta/2$, then

$$\delta/2\leqslant\varphi(x)\leqslant 1 \qquad \text{for} \quad |x|\leqslant d-\delta d \tag{14}$$

$$-1\leqslant\varphi(x)\leqslant-\delta/2 \qquad \text{for} \quad |x|\geqslant d+\delta d. \tag{15}$$

By (12) and (14) we get $0\leqslant 1-\sigma(x)\leqslant\gamma$ for $|x|\leqslant d-\delta d$, i.e., σ satisfies (5). By (11) and (15) we obtain

$$0\leqslant\sigma(x)\leqslant(1+\varphi(x))^{2r}\gamma=\left(\frac{2d^2}{d^2+x^2}\right)^{2r}\gamma\leqslant\left(\frac{2d}{d+x}\right)^{4r}\gamma$$

for $|x|\geqslant d+\delta d$, i.e., σ satisfies (6). ∎

Proof of Lemma 3.1. Without loss of generality we shall assume that $\Delta = [-b, b]$. If $\lambda \geqslant 1$, then the rational function $R \equiv 0$ satisfies the requirements of Lemma 3.1.

Let $0 < \lambda < 1$. Consider the rational function

$$R = \sigma Q,$$

where σ is the rational function from Lemma 3.3 with $\delta = \lambda^p/2$, $d = b/(1+\delta)$, $\gamma = \lambda$, $r = k+1$.

By (4) we get

$$\deg R \leqslant \deg \sigma + \deg Q \leqslant B \ln\left(e + \frac{1}{\delta}\right) \ln\left(e + \frac{1}{\gamma}\right) + 4r + k$$
$$\leqslant B \ln\left(e + \frac{2}{\lambda^p}\right) \ln\left(e + \frac{1}{\lambda}\right) + 4(k+1) + k$$

and hence

$$\deg R \leqslant B_1 \ln^2\left(e + \frac{1}{\lambda}\right), \qquad B_1 = B_1(p, k). \tag{16}$$

Now we estimate $\|Q - R\|_{L_p(\Delta)}$. By (5) we obtain

$$|Q(x) - R(x)| = (1 - \sigma(x))|Q(x)| \leqslant \lambda |Q(x)| \quad \text{for } |x| \leqslant d - \delta d = \frac{1-\delta}{1+\delta} b. \tag{17}$$

If $d - \delta d \leqslant |x| \leqslant d + \delta d$, i.e., $((1-\delta)/(1+\delta))\, b \leqslant |x| \leqslant b$, then by (7) and Lemma 3.2 we get

$$|Q(x) - R(x)| \leqslant |Q(x)| \leqslant C(k) \frac{\|Q\|_{L_p(\Delta)} |x|^k}{\left(2 \frac{1-\delta}{1+\delta} b\right)^{k+1/p}} \leqslant C_1(k) \frac{\|Q\|_{L_p(\Delta)}}{|\Delta|^{1/p}}. \tag{18}$$

Using (17) and (18) we obtain

$$\|Q - R\|_{L_p(\Delta)} = \left(\int_{-b}^{b} |Q(x) - R(x)|^p dx\right)^{1/p}$$
$$= \left(\int_{-b}^{-((1-\delta)/(1+\delta))b} + \int_{-((1-\delta)/(1+\delta))b}^{((1-\delta)/(1+\delta))b} + \int_{((1-\delta)/(1+\delta))b}^{b}\right)^{1/p}$$
$$\leqslant \left\{2\left(b - \frac{1-\delta}{1+\delta} b\right)\left(C_1 \frac{\|Q\|_{L_p(\Delta)}}{|\Delta|^{1/p}}\right)^p + \int_{-b}^{b} \lambda^p |Q(x)|^p dx\right\}^{1/p}$$
$$\leqslant C_2 \lambda \|Q\|_{L_p(\Delta)}.$$

Hence

$$\|Q-R\|_{L_p(\Delta)}\leqslant C_2\lambda\|Q\|_{L_p(\Delta)}, \qquad C_2=C_2(p,k). \tag{19}$$

If $|x|>b$, using (6), Lemma 3.2 and the fact that $4r\geqslant k+4$ we obtain

$$\begin{aligned}|R(x)|=|\sigma(x)\,Q(x)|&\leqslant C\frac{\|Q\|_{L_p(\Delta)}|x|^k}{|\Delta|^{k+1/p}}\left(\frac{2d}{d+|x|}\right)^{4r}\gamma\\&\leqslant C\frac{\|Q\|_{L_p(\Delta)}|x|^k}{|\Delta|^{k+1/p}}\left(\frac{2b}{b+|x|}\right)^{k+4}\\&=C\left(\frac{|\Delta|}{|\Delta|+|x|-b}\right)^4\frac{\lambda\|Q\|_{L_p(\Delta)}}{|\Delta|^{1/p}}.\end{aligned}$$

Hence

$$|R(x)|\leqslant C(k)\left(\frac{|\Delta|}{|\Delta|+\rho(x,\Delta)}\right)^4\frac{\lambda\|Q\|_{L_p(\Delta)}}{|\Delta|^{1/p}} \qquad \text{for} \quad |x|>b. \tag{20}$$

The estimates (16), (19), and (20) prove Lemma 3.1. ∎

Proof of Theorem 3.1. Assume that $\varphi\in S(k,m,[a,b])$, $k\geqslant 1$, $m\geqslant 1$. Then there are points $x_0=a<x_1<\cdots<x_m=b$ such that for each i $(1\leqslant i\leqslant m)$ there is polynomial $Q_i\in H_{k-1}$ such that $\varphi(x)=Q_i(x)$ for $x\in(x_{i-1},x_i)$. We put $\varphi(x)=0$ for $x\in(-\infty,\infty)\setminus[a,b]$. Let $\lambda>0$ and $1\leqslant p<\infty$.

In what follows we shall use the following notations: $\Delta=[a,b]$,

$$\Delta_0=(-\infty,x_0], \qquad \Delta_i=[x_{i-1},x_i], \qquad i=1,2,\dots,m,$$

$$\Delta_{m+1}=[x_m,\infty), \qquad A=(\|\varphi\|_{L_p(\Delta)}/m)^{1/p}.$$

Without loss of generality we shall assume that $\|\varphi\|_{L_p(\Delta_i)}\neq 0$ for $i=1,2,\dots,m$.

Now we apply Lemma 3.1 for the function φ in each interval Δ_i $(1\leqslant i\leqslant m)$ with $\lambda_i=\lambda A/\|\varphi\|_{L_p(\Delta_i)}$. We obtain that for each i $(1\leqslant i\leqslant m)$ there is a rational function R_i such that

$$\deg R_i\leqslant D\ln^2\left(e+\frac{1}{\lambda_i}\right)=D\ln^2\left(e+\frac{\|\varphi\|_{L_p(\Delta_i)}}{\lambda A}\right), \tag{21}$$

$$\|\varphi-R_i\|_{L_p(\Delta_i)}\leqslant D\lambda_i\|\varphi\|_{L_p(\Delta_i)}=D\lambda A \tag{22}$$

and

$$|R_i(x)|\leqslant D\left(\frac{|\Delta_i|}{|\Delta_i|+\rho(x,\Delta_i)}\right)^4\frac{\lambda A}{|\Delta_i|^{1/p}} \qquad \text{for} \quad x\in(-\infty,\infty)\setminus\Delta_i, \tag{23}$$

where $D=D(p,k)>0$.

We shall show that the rational function $R=\sum_{i=1}^{m} R_i$ satisfies the requirements of Theorem 3.2. First we estimate deg R. To this end we use (21) and the facts that the function $\ln^2(e+x)$ is concave on $[0, \infty)$ and $\ln^2(e+x) < 4\ln^2(e+x^p)$ for $x>0$. We get

$$\begin{aligned}
\deg R &\leqslant \sum_{i=1}^{m} \deg R_i \leqslant \sum_{i=1}^{m} D \ln^2\left(e+\frac{\|\varphi\|_{L_p(\Delta_i)}}{\lambda A}\right) \\
&\leqslant 4D \sum_{i=1}^{m} \ln^2\left(e+\frac{\|\varphi\|^p_{L_p(\Delta_i)}}{\lambda^p A^p}\right) \\
&\leqslant 4Dm \ln^2\left(e+\frac{\sum_{i=1}^{m} \|\varphi\|^p_{L_p(\Delta_i)}}{m\lambda^p A^p}\right) \\
&= 4Dm \ln^2\left(e+\frac{\|\varphi\|^p_{L_p(\Delta)}}{m\lambda^p A^p}\right) \\
&= 4Dm \ln^2\left(e+\frac{1}{\lambda^p}\right) \leqslant 4Dpm \ln^2\left(e+\frac{1}{\lambda}\right).
\end{aligned}$$

Thus we have

$$\deg R \leqslant D_1 m \ln^2\left(e+\frac{1}{\lambda}\right), \qquad D_1 = D_1(p,k). \tag{24}$$

It remains to estimate $\|\varphi - R\|_{L_p(-\infty,\infty)}$. We need some notations.

DEFINITION 3.1. We shall call the set of intervals $\{\Delta_\nu\colon i_0 \leqslant \nu \leqslant i_1\}$, $1 \leqslant i_0 \leqslant i_1 \leqslant m+1$, a *left class of interval* or briefly a *left class*, if $|\Delta_\nu| < |\Delta_{i_1}|$ for $\nu = i_0, i_0+1, \ldots, i_1-1$ and $|\Delta_{i_0-1}| \geqslant |\Delta_{i_1}|$.

We shall suppose that $|\Delta_0| = |\Delta_{m+1}| > |\Delta_\nu|$, $\nu = 1, 2, \ldots, m$.

By Ω we shall denote the set of all left classes of intervals.

Some Properties of the left classes of intervals. (a) If K, $\tilde{K} \in \Omega$, then $K \cap \tilde{K} = \varnothing$ or $K \subset \tilde{K}$ or $\tilde{K} \subset K$. Therefore the relation $K \subset \tilde{K}$ realizes order in the set Ω with the final element the left class $\{\Delta_\nu\colon 1 \leqslant \nu \leqslant m+1\}$.

(b) For each i $(1 \leqslant i \leqslant m+1)$ there is exactly one left class $K_i \in \Omega$ such that the interval Δ_i is the last interval is K_i, i.e., $K_i = \{\Delta_\nu\colon i_0 \leqslant \nu \leqslant i\}$. In what follows we shall use this notation. Thus there is one to one mapping of the set $\{\Delta_\nu\colon 1 \leqslant \nu \leqslant m+1\}$ on the set Ω. Consequently, the number of the elements of Ω is $m+1$.

(c) If $K \in \Omega$ and $\Delta_i \in K$ $(1 \leqslant i \leqslant m+1)$, then $K_i \subset K$.

DEFINITION 3.2. We shall call the left class $\tilde{K}$ *left subclass of first order of the left class* K, if $\tilde{K} \subset K$, $\tilde{K} \neq K$ and there is no class $K^* \in \Omega$, $K^* \neq K$ such that $\tilde{K} \subset K^* \subset K$, i.e., K follows $\tilde{K}$ immediately.

By Ω_i $(1 \leqslant i \leqslant m+1)$ we shall denote the set of all subclasses of first order of the left class K_i and by μ_i the number of the elements of Ω_i.

(d) We have for $i = 1, 2,..., m+1$

$$K_i = \bigcup_{K \in \Omega_i} K \cup \{\Delta_i\}.$$

More exactly, for each i $(1 \leqslant i \leqslant m+1)$ there are indexes $0 \leqslant j_0 < j_1 < \cdots < j_{\mu_i} = i-1$ so that

$$K_i = \{\Delta_s : j_0 + 1 \leqslant s \leqslant i\} = \bigcup_{\nu=1}^{\mu_i} K_{j_\nu} \cup \{\Delta_i\},$$

$$\Omega_i = \{K_{j_\nu} : 1 \leqslant \nu \leqslant \mu_i\},$$

$$K_{j_\nu} = \{\Delta_s : j_{\nu-1} + 1 \leqslant s \leqslant j_\nu\}$$

Hence $|\Delta_{j_0}| \geqslant |\Delta_i| > |\Delta_{j_1}| \geqslant |\Delta_{j_2}| \geqslant \cdots \geqslant |\Delta_{j_{\mu_i}}|$ and $|\Delta_s| < |\Delta_{j_\nu}|$ for $s = j_{\nu-1}+1, j_{\nu-1}+2,..., j_\nu - 1$; $\nu = 1, 2,..., \mu_i$.

(e) Each class $K \in \Omega$, $K \neq K_{m+1} = \{\Delta_\nu : 1 \leqslant \nu \leqslant m+1\}$, is a left subclass of first order of some left class and therefore $\Omega = \bigcup_{i=1}^{m+1} \Omega_i \cup \{K_{m+1}\}$. On the otherhand $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$. Consequently,

$$\sum_{i=1}^{m+1} \mu_i = m. \tag{25}$$

The properties (a) – (e) of the left classes follow immediately by the definitions.

Analogously (more exactly symmetrically) we introduce the notions *right class of intervals* and *right subclass of first order of some right class*. We shall denote by Ω^* the set of all right classes, by K_i^* the right class in which Δ_i is the first interval $(0 \leqslant i \leqslant m)$, by Ω_i^* the set of all right subclass of first of K_i^* and by μ_i^* the number of the elements in Ω_i^*. The right classes have properties symmetrical to the properties (a)–(e). We formulate only the following property:

$$\sum_{i=0}^{m} \mu_i^* = m. \tag{26}$$

We need the following lemmas, where we use the notations introduced above.

LEMMA 3.4. *The following inequality holds true for* $1 \leqslant i \leqslant m$:

$$\left|\sum_{\Delta_v \in K_i} R_v(x)\right| \leqslant C\left(\frac{|\Delta_i|}{|\Delta_i| + x - x_i}\right)^3 \frac{\lambda A}{|\Delta_i|^{1/p}} \quad \text{for} \quad x \geqslant x_i, \tag{27}$$

where $C = C(p, k)$.

Proof. Let $K_i = \{\Delta_v : i_0 \leqslant v \leqslant i\}$. If $i_0 = i$, then the estimate (27) follows by (23) immediately.

Let $i_0 < i$ and $x \geqslant x_i$. By (23) we obtain

$$\left|\sum_{\Delta_v \in K_i} R_v(x)\right| \leqslant D\lambda A \sum_{v=i_0}^{i} \left(\frac{|\Delta_v|}{\sum_{s=v}^{i} |\Delta_s| + x - x_i}\right)^4 \frac{1}{|\Delta_v|^{1/p}}.$$

By the definition of left class it follows that $|\Delta_v| < |\Delta_i|$, $v = i_0, i_0 + 1, ..., i - 1$. Denote

$$G_{-r} = \{v : 2^{-r}|\Delta_i| < |\Delta_v| \leqslant 2^{-r+1}|\Delta_i|\}, \qquad r = 1, 2, ...$$

Clearly, for $r \geqslant 1$

$$\begin{aligned}
&\sum_{v \in G_{-r}} \left(\frac{|\Delta_v|}{\sum_{s=v}^{i} |\Delta_s| + x - x_i}\right)^4 \frac{1}{|\Delta_v|^{1/p}} \\
&\quad \leqslant \sum_{s=1}^{\infty} \left(\frac{2^{-r+1}|\Delta_i|}{s2^{-r}|\Delta_i| + |\Delta_i| + x - x_i}\right)^4 \frac{1}{(2^{-r}|\Delta_i|)^{1/p}} \\
&\quad \leqslant \frac{2^4}{2^{-r/p}|\Delta_i|^{1/p}} \int_0^{\infty} \left(\frac{2^{-r}|\Delta_i|}{t \cdot 2^{-r}|\Delta_i| + |\Delta_i| + x - x_i}\right)^4 dt \\
&\quad \leqslant 2^5 \cdot 2^{-(3-1/p)r} \left(\frac{|\Delta_i|}{|\Delta_i| + x - x_i}\right)^3 \frac{1}{|\Delta_i|^{1/p}}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\left|\sum_{\Delta_v \in K_i} R_v(x)\right| &\leqslant D\lambda A \sum_{r=1}^{\infty} \sum_{v \in G_{-r}} \left(\frac{|\Delta_v|}{\sum_{s=v}^{i} |\Delta_s| + x - x_i}\right)^4 \frac{1}{|\Delta_v|^{1/p}} \\
&\leqslant 2^5 D\lambda A \left(\frac{|\Delta_i|}{|\Delta_i| + x - x_i}\right)^3 \frac{1}{|\Delta_i|^{1/p}} \sum_{r=1}^{\infty} 2^{-(3-1/p)r} \\
&\leqslant C\left(\frac{|\Delta_i|}{|\Delta_i| + x - x_i}\right)^3 \frac{1}{|\Delta_i|^{1/p}}, \qquad C = C(p, k). \quad \blacksquare
\end{aligned}$$

The importance of the notations introduced will become clear by the following lemma.

LEMMA 3.5. *The following estimate holds true for* $i = 2, 3, \ldots, m+1$:

$$\int_{\Delta_i} \left| \sum_{\nu=1}^{i-1} R_\nu(x) \right|^p dx \leqslant C(\mu_i + 1)\, \lambda^p A^p,$$

where $C = C(p, k)$.

Proof. Let $2 \leqslant i \leqslant m+1$. By the property (d) of the left classes of intervals it follows that there are indexes $0 \leqslant j_0 < j_1 < \cdots < j_{\mu_i} = i-1$ so that

$$\begin{aligned} K_i &= \{\Delta_s : j_0 + 1 \leqslant s \leqslant i\} = \bigcup_{\nu=1}^{\mu_i} K_{j_\nu} \cup \{\Delta_i\}, \\ K_{j_\nu} &= \{\Delta_s : j_{\nu-1} + 1 \leqslant s \leqslant j_\nu\}, \\ \Omega_i &= \{K_{j_\nu} : 1 \leqslant \nu \leqslant \mu_i\}. \end{aligned}$$

Hence

$$\begin{aligned} &|\Delta_{j_0}| \geqslant |\Delta_i| > |\Delta_{j_1}| \geqslant |\Delta_{j_2}| \geqslant \cdots \geqslant |\Delta_{j_{\mu_i}}|, \ |\Delta_s| < |\Delta_{j_\nu}| \\ &\text{for} \quad s = j_{\nu-1} + 1, j_{\nu-1} + 2, \ldots, j_\nu - 1;\ \nu = 1, 2, \ldots, \mu_i. \end{aligned} \tag{28}$$

We have

$$\begin{aligned} &\int_{\Delta_i} \left| \sum_{\nu=1}^{i-1} R_\nu(x) \right|^p dx \\ &\quad \leqslant 2^{p-1} \int_{\Delta_i} \left| \sum_{\nu=1}^{j_0} R_\nu(x) \right|^p dx + 2^{p-1} \int_{\Delta_i} \left| \sum_{\nu=j_0+1}^{i-1} R_\nu(x) \right|^p dx \\ &\quad = I_1 + I_2. \end{aligned}$$

First we estimate I_1 using (23) and the fact that $|\Delta_{j_0}| \geqslant |\Delta_i|$; see (28). We obtain for $x \in \Delta_i$

$$\begin{aligned} \left| \sum_{\nu=1}^{j_0} R_\nu(x) \right| &\leqslant \sum_{\nu=1}^{j_0} D \left(\frac{|\Delta_\nu|}{|\Delta_\nu| + x - x_\nu} \right)^4 \frac{\lambda A}{|\Delta_\nu|^{1/p}} \\ &\leqslant D\lambda A \sum_{\nu=1}^{j_0} \left(\frac{|\Delta_\nu|}{\sum_{s=\nu}^{j_0} |\Delta_s| + x - x_{i-1}} \right)^4 \frac{1}{|\Delta_\nu|^{1/p}}. \end{aligned}$$

Denote

$$G_r = \{\nu : 2^r |\Delta_{j_0}| < |\Delta_\nu| \leqslant 2^{r+1} |\Delta_{j_0}|,\ 1 \leqslant \nu \leqslant j_0\},$$
$$r = 0, \pm 1, \pm 2, \ldots.$$

Clearly, for $r \geqslant 0$, $x \in \Delta_i$ we have

$$\sum_{\nu \in G_r} \left(\frac{|\Delta_\nu|}{\sum_{s=\nu}^{j_0} |\Delta_s| + x - x_{i-1}} \right)^4 \frac{1}{|\Delta_\nu|^{1/p}}$$
$$\leqslant \sum_{l=1}^{\infty} \left(\frac{2^{r+1}|\Delta_{j_0}|}{l \cdot 2^r |\Delta_{j_0}| + x - x_{i-1}} \right)^4 \frac{1}{(2^r |\Delta_{j_0}|)^{1/p}}$$
$$\leqslant \frac{2^4}{2^{r/p} |\Delta_{j_0}|^{1/p}} \left\{ \left(\frac{2^r |\Delta_{j_0}|}{2^r |\Delta_{j_0}| + x - x_{i-1}} \right)^4 + \int_1^\infty \left(\frac{2^r |\Delta_{j_0}|}{t \cdot 2^r |\Delta_{j_0}| + x - x_{i-1}} \right)^4 dt \right\}$$
$$\leqslant \frac{2^5}{2^{r/p} |\Delta_{j_0}|^{1/p}}.$$

If $r \leqslant -1$, then

$$\sum_{\nu \in G_r} \left(\frac{|\Delta_\nu|}{\sum_{s=\nu}^{j_0} |\Delta_s| + x - x_{i-1}} \right)^4 \frac{1}{|\Delta_\nu|^{1/p}}$$
$$\leqslant \sum_{s=1}^{\infty} \left(\frac{2^{r+1}|\Delta_{j_0}|}{s \cdot 2^r |\Delta_{j_0}| + |\Delta_{j_0}| + x - x_{i-1}} \right)^4 \frac{1}{(2^r |\Delta_{j_0}|)^{1/p}}$$
$$\leqslant \frac{2^4}{2^{r/p} |\Delta_{j_0}|^{1/p}} \int_0^\infty \left(\frac{2^r |\Delta_{j_0}|}{t \cdot 2^r |\Delta_{j_0}| + |\Delta_{j_0}| + x - x_{i-1}} \right)^4 dt$$
$$\leqslant \frac{2^3}{2^{r/p} |\Delta_{j_0}|^{1/p}} \left(\frac{2^r |\Delta_{j_0}|}{|\Delta_{j_0}| + x - x_{i-1}} \right)^3 \leqslant \frac{2^3 \cdot 2^{(3-1/p)r}}{|\Delta_{j_0}|^{1/p}}.$$

Consequently

$$\left| \sum_{\nu=1}^{j_0} R_\nu(x) \right| \leqslant D\lambda A \sum_{r=-\infty}^{\infty} \sum_{\nu \in G_r} \left(\frac{|\Delta_\nu|}{\sum_{s=\nu}^{j_0} |\Delta_s| + x - x_{i-1}} \right)^4 \frac{1}{|\Delta_\nu|^{1/p}}$$
$$\leqslant D\lambda A \left\{ \sum_{r=0}^{\infty} \frac{2^5}{2^{r/p} |\Delta_{j_0}|^{1/p}} + \sum_{r=1}^{\infty} \frac{2^3}{2^{(3-1/p)r} |\Delta_{j_0}|^{1/p}} \right\}$$
$$\leqslant C_1 \frac{\lambda A}{|\Delta_{j_0}|^{1/p}}, \qquad C_1 = C_1(p, k).$$

Integrating we obtain

$$I_1 = 2^{p-1} \int_{\Delta_i} \left| \sum_{\nu=1}^{j_0} R_\nu(x) \right|^p dx \leqslant 2^{p-1} |\Delta_i| \frac{C_1^p \lambda^p A^p}{|\Delta_{j_0}|} \leqslant C_2 \lambda^p A^p, \tag{29}$$

where $C_2 = C_2(p, k)$.

Now we estimate I_2. Using Lemma 3.4 we obtain

$$\left|\sum_{v=j_0+1}^{i-1} R_v(x)\right| \leqslant \sum_{s=1}^{\mu_i} \left|\sum_{v=j_{s-1}+1}^{j_s} R_v(x)\right| \leqslant C \sum_{s=1}^{\mu_i} \left(\frac{|\Delta_{j_s}|}{|\Delta_{j_s}|+x-x_{j_s}}\right)^3 \frac{\lambda A}{|\Delta_{j_s}|^{1/p}}. \quad (30)$$

Denote $y_l = x_{i-1} + \sum_{v=l}^{\mu_i} |\Delta_{j_v}|$, $l = 1, 2, ..., \mu_i$, and $y_{\mu_i+1} = x_{i-1}$. We have $x_{i-1} = y_{\mu_i+1} < y_{\mu_i} < \cdots < y_1$.

If $x \in [y_1, \infty)$, then by (28) and (30) we get

$$\left|\sum_{v=j_0+1}^{i-1} R_v(x)\right| \leqslant C \sum_{s=1}^{\mu_i} \left(\frac{|\Delta_{j_s}|}{\sum_{r=1}^{\mu_i} |\Delta_{j_r}| + x - y_1}\right)^3 \frac{\lambda A}{|\Delta_{j_s}|^{1/p}}$$
$$\leqslant C\lambda A \sum_{s=1}^{\mu_i} \left(\frac{|\Delta_{j_s}|}{s|\Delta_{j_s}| + x - y_1}\right)^3 \frac{1}{|\Delta_{j_s}|^{1/p}}.$$

Hence

$$\left(\int_{y_1}^{\infty} \left|\sum_{v=j_0+1}^{i-1} R_v(x)\right|^p dx\right)^{1/p}$$
$$\leqslant C\lambda A \left\{\int_{y_1}^{\infty} \left(\sum_{s=1}^{\mu_i} \left(\frac{|\Delta_{j_s}|}{s\,|\Delta_{j_s}| + x - y_1}\right)^3 \frac{1}{|\Delta_{j_s}|^{1/p}}\right)^p dx\right)^{1/p}$$
$$\leqslant C\lambda A \sum_{s=1}^{\mu_i} \left\{\int_{y_1}^{\infty} \left(\frac{|\Delta_{j_s}|}{s\,|\Delta_{j_s}| + x - y_1}\right)^{3p} \frac{1}{|\Delta_{j_s}|}\, dx\right\}^{1/p}$$
$$\leqslant C\lambda A \sum_{s=1}^{\infty} \frac{1}{s^{3-1/p}}.$$

Consequently

$$\int_{y_1}^{\infty} \left|\sum_{v=j_0+1}^{i-1} R_v(x)\right|^p dx \leqslant C_3 \lambda^p A^p, \qquad C_3 = C_3(p, k). \quad (31)$$

Let $x \in [y_{l+1}, y_l]$, $1 \leqslant l \leqslant \mu_i$. By (30) we obtain

$$\left|\sum_{v=j_0+1}^{i-1} R_v(x)\right| \leqslant C\lambda A \left\{\sum_{s=1}^{l} \left(\frac{|\Delta_{j_s}|}{\sum_{v=s}^{l} |\Delta_{j_v}| + x - y_{l+1}}\right)^3 \frac{1}{|\Delta_{j_s}|^{1/p}}\right.$$
$$\left. + \sum_{s=l+1}^{\mu_i} \left(\frac{|\Delta_{j_s}|}{\sum_{v=l+1}^{\mu_i} |\Delta_{j_v}| + x - y_{l+1}}\right)^3 \frac{1}{|\Delta_{j_s}|^{1/p}}\right\}$$
$$= C\lambda A\{\sigma_1 + \sigma_2\}.$$

First we estimate σ_1. Denote

$$G_r = \{s: 2^r|\Delta_{j_l}| \leqslant |\Delta_{j_s}| < 2^{r+1}|\Delta_{j_l}|,\ 1 \leqslant s \leqslant l\}, \qquad r = 0, 1, ...$$

Using (28) we obtain

$$\sigma_1 \leqslant \sum_{r=0}^{\infty} \sum_{s \in G_r} \left(\frac{|\Delta_{j_s}|}{\sum_{v=s}^{l} |\Delta_{j_v}| + x - y_{l+1}} \right)^3 \frac{1}{|\Delta_{j_s}|^{1/p}}$$
$$\leqslant \sum_{r=0}^{\infty} \sum_{m=1}^{\infty} \left(\frac{2^{r+1}|\Delta_{j_l}|}{m \cdot 2^r |\Delta_{j_l}| + x - y_{l+1}} \right)^3 \frac{1}{2^r |\Delta_{j_l}|^{1/p}}$$
$$\leqslant \sum_{r=0}^{\infty} \frac{2^4}{2^{r/p} |\Delta_{j_l}|^{1/p}} \leqslant \frac{C_4(p)}{|\Delta_{j_l}|^{1/p}}.$$

Using (28) again we get

$$\sigma_2 \leqslant \sum_{s=l+1}^{\mu_i} \left(\frac{|\Delta_{j_s}|}{(s-l)|\Delta_{j_s}| + x - y_{l+1}} \right)^3 \frac{1}{|\Delta_{j_s}|^{1/p}}.$$

Consequently

$$\left| \sum_{v=j_0+1}^{i-1} R_v(x) \right| \leqslant C_5 \lambda A \left\{ \frac{1}{|\Delta_{j_l}|^{1/p}} + \sum_{s=l+1}^{\mu_i} \left(\frac{|\Delta_{j_s}|}{(s-l)|\Delta_{j_s}| + x - y_{l+1}} \right)^3 \frac{1}{|\Delta_{j_s}|^{1/p}} \right\},$$

where $C_5 = C_5(p, k)$. Now we take the L_p norm and obtain

$$\left(\int_{y_{l+1}}^{y_l} \left| \sum_{v=j_0+1}^{i-1} R_v(x) \right|^p dx \right)^{1/p}$$
$$\leqslant C_5 \lambda A \left\{ \left(\int_{y_{l+1}}^{y_l} \frac{dx}{|\Delta_{j_l}|} \right)^{1/p} \right.$$
$$\left. + \sum_{s=l+1}^{\mu_i} \left(\int_{y_{l+1}}^{y_l} \left(\frac{|\Delta_{j_s}|}{(s-l)|\Delta_{j_s}| + x - y_{l+1}} \right)^{3p} \frac{dx}{|\Delta_{j_s}|} \right)^{1/p} \right\}$$
$$\leqslant C_5 \lambda A \left\{ 1 + \sum_{s=l+1}^{\infty} \frac{1}{(s-l)^{3-1/p}} \right\}.$$

Consequently, for $l = 1, 2, \ldots, \mu_i$

$$\int_{y_{l+1}}^{y_l} \left| \sum_{v=j_0+1}^{i-1} R_v(x) \right|^p dx \leqslant C_6 \lambda^p A^p, \qquad C_6 = C_6(p, k).$$

Combining this estimate with (31) we get

$$\int_{\Delta_i} \left| \sum_{v=1}^{i-1} R_v(x) \right|^p dx \leqslant \int_{x_{i-1}}^{\infty} \left| \sum_{v=1}^{i-1} R_v(x) \right|^p dx$$
$$\leqslant C(\mu_i + 1)\, \lambda^p A^p, \qquad C = C(p, k). \quad \blacksquare$$

The following lemma can be proved in a simalar (symmetrical) way.

LEMMA 3.6. *The following estimate holds true for* $i=0, 1, \ldots, m-1$:

$$\int_{\Delta_i} \left| \sum_{\nu=i+1}^{m} R_\nu(x) \right|^p dx \leqslant C(\mu_i^* + 1)\, \lambda^p A^p,$$

where $C = C(p, k)$.

Completion of the Proof of Theorem 3.1. Using (22), (25), (26), and Lemmas 3.5 and 3.6 we obtain

$$\begin{aligned}
\|\varphi - R\|_{L_p(-\infty,\infty)} &= \left\| \varphi - \sum_{\nu=1}^{m} R_\nu \right\|_{L_p(-\infty,\infty)} = \left(\sum_{i=0}^{m+1} \left\| \varphi - \sum_{\nu=1}^{m} R_\nu \right\|_{L_p(\Delta_i)}^p \right)^{1/p} \\
&\leqslant \left\{ \sum_{i=0}^{m+1} 3^{p-1} \left(\int_{\Delta_i} \left| \sum_{\nu<i} R_\nu(x) \right|^p dx \right.\right. \\
&\quad \left.\left. + \int_{\Delta_i} |\varphi(x) - R_i(x)|^p dx + \int_{\Delta_i} \left| \sum_{\nu>i} R_\nu(x) \right|^p dx \right) \right\}^{1/p} \\
&\leqslant 3 \left\{ C(\mu_0^* + 1)\, \lambda^p A^p + \sum_{i=1}^{m} \left(C(\mu_i + 1)\, \lambda^p A^p + D^p \lambda^p A^p \right.\right. \\
&\quad \left.\left. + C(\mu_i^* + 1)\, \lambda^p A^p \right) + C(\mu_{m+1} + 1)\, \lambda^p A^p \right\}^{1/p} \\
&\leqslant C_1 \lambda A \left(\sum_{i=1}^{m+1} \mu_i + m + \sum_{i=0}^{m} \mu_i^* \right)^{1/p} \\
&\leqslant 3C_1 m^{1/p} \lambda A = 3C_1 \lambda \|\varphi\|_{L_p(\Delta)}.
\end{aligned}$$

Consequently

$$\|\varphi - R\|_{L_p(-\infty,\infty)} \leqslant C\lambda \|\varphi\|_{L_p[a,b]}, \qquad C = C(p, k).$$

This estimate together with (24) establishes Theorem 3.1. ∎

Proof of Theorem 2.1. Let $\varphi_m \in S(k, m, [a, b])$ and φ_m satisfies $\|f - \varphi_m\|_{L_p(\Delta)} = S_m^k(f)_p$, $m \geqslant 1$, $\Delta = [a, b]$. For $i \geqslant 1$ we have $\varphi_{2^i} - \varphi_{2^{i-1}} \in S(k, 2^{i+1}, \Delta)$ and

$$\|\varphi_{2^i} - \varphi_{2^{i-1}}\|_{L_p(\Delta)} \leqslant \|f - \varphi_{2^i}\|_{L_p(\Delta)} + \|f - \varphi_{2^{i-1}}\|_{L_p(\Delta)} \leqslant 2S_{2^{i-1}}^k(f)_p.$$

Using Theorem 3.1 and the function $\varphi_{2^i} - \varphi_{2^{i-1}}$ with $\lambda = 2^{(i-s)\alpha}$ we obtain that there is a rational function R_i such that

$$\deg R_i \leqslant D 2^{i+1} \ln^2(e + 2^{(s-i)\alpha}), \qquad D = D(p, k) > 1, \tag{32}$$

and

$$\|\varphi_{2^i} - \varphi_{2^{i-1}} - R_i\|_{L_p(\Delta)} \leqslant 2^{(i-s)\alpha} \|\varphi_{2^i} - \varphi_{2^{i-1}}\|_{L_p(\Delta)} \leqslant 2^{(i-s)\alpha+1} S^k_{2^{i-1}}(f)_p. \tag{33}$$

Let $s \geqslant 0$ be an integer. Consider the rational function $R = \sum_{i=0}^{s} R_i$, where $R_0 = \varphi_1 \in H_{k-1}$. First we estimate deg R. By (32) we obtain

$$N = \deg R \leqslant \sum_{i=0}^{s} \deg R_i \leqslant \sum_{i=0}^{s} D \cdot 2^{i+1} \ln^2(e + 2^{(s-i)\alpha}) + k - 1$$
$$\leqslant D_1(\alpha+1)^2 \sum_{i=1}^{s} 2^i (s-i)^2 \leqslant D_2(\alpha+1)^2 \cdot 2^s.$$

Consequently

$$N = \deg R \leqslant D_2(\alpha+1)^2 2^s, \qquad D_2 = D_2(p, k). \tag{34}$$

Now estimate $\|f - R\|_{L_p(\Delta)}$. By (33) we get

$$\|f - R\|_{L_p(\Delta)}$$
$$\leqslant \|f - \varphi_{2^s}\|_{L_p(\Delta)} + \sum_{i=1}^{s} \|\varphi_{2^i} - \varphi_{2^{i-1}} - R_i\|_{L_p(\Delta)} + \|\varphi_1 - R_0\|_{L_p(\Delta)}$$
$$\leqslant S^k_{2^s}(f)_p + \sum_{i=1}^{s} 2^{(i-s)\alpha+1} S^k_{2^{i-1}}(f)_p$$
$$\leqslant 2^{\alpha+1} \cdot 2^{-s\alpha} \sum_{i=0}^{s} 2^{i\alpha} S^k_{2^i}(f)_p$$
$$\leqslant 2^{2\alpha+1} \cdot 2^{-s\alpha} \sum_{\nu=1}^{2^s} \nu^{\alpha-1} S^k_\nu(f)_p.$$

Hence, by (34) and the estimates above it follows that for each $s \geqslant 0$

$$R_N(f, \Delta)_p \leqslant 2^{2\alpha+1} \cdot 2^{-s\alpha} \sum_{\nu=1}^{2^s} \nu^{\alpha-1} S^k_\nu(f)_p, \tag{35}$$

where $N \leqslant D_2(\alpha+1)^2 \cdot 2^s$, $D_2 = D_2(p, k) > 1$.

Now let $n \geqslant \max\{1, k-1\}$. If $n \leqslant A = D_2(\alpha+1)^2$, then

$$R_n(f)_p \leqslant R_{k-1}(f)_p \leqslant S^k_1(f)_p \leqslant A^\alpha n^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1} S^k_\nu(f)_p.$$

Consequently

$$R_n(f)_p \leqslant C_1 n^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1} S_\nu^k(f)_p \qquad \text{for} \quad \max\{1, k-1\} \leqslant n \leqslant A, \tag{36}$$

where $C_1 = C_1(p, k)$.

Let $n > A$. We choose $s \geqslant 0$ such that $A \cdot 2^s < n \leqslant A \cdot 2^{s+1}$. Then by (35) we obtain

$$R_n(f)_p \leqslant 2^{2\alpha+1} \cdot 2^{-s\alpha} \sum_{\nu=1}^{2^s} \nu^{\alpha-1} S_\nu^k(f)_p$$

$$\leqslant C_2(p, k, \alpha)\, n^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1} S_\nu^k(f)_p.$$

These estimates together with (36) imply (2). The estimate (3) can be proved in a similar way. ▮

Proof of Corollary 2.2. Since $\omega(2\delta) \leqslant 2\omega(\delta)$ for $\delta \geqslant 0$, then $\omega(\lambda\delta) \leqslant 2(\lambda+1)\,\omega(\delta)$ for $\lambda, \delta \geqslant 0$. Now if $f \in L_p[a, b]$ and $S_n^k(f)_p \leqslant C_1 n^{-\gamma} \omega(n^{-1})$, $1 \leqslant p < \infty$, then using Theorem 2.1 we obtain

$$R_n(f)_p \leqslant C n^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1} C_1 \nu^{-\gamma} \omega(\nu^{-1})$$

$$\leqslant C_2 n^{-\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1-\gamma} \left(\frac{n}{\nu}+1\right) \omega(n^{-1})$$

$$= O(n^{-\gamma} \omega(n^{-1}))$$

for $\alpha \geqslant \gamma + 2$. ▮

Remark 3.1. The proof of Theorem 2.1 is based on Lemma 3.3. Clearly, this lemma can be used successfuly for the uniform rational approximation of functions with finite support on $(-\infty, \infty)$. It is easy to see that Lemma 3.3 implies the following estimate: If

$$\varphi(x) = 1 - |x| \qquad \text{for} \quad |x| \leqslant 1,$$

$$\varphi(x) = 0 \qquad \text{for} \quad |x| > 1$$

then

$$R_n(\varphi, (-\infty, \infty))_C = O(e^{-C\sqrt{n}}), \qquad C > 0, \tag{37}$$

where

$$R_n(\varphi, (-\infty, \infty))_C = \inf_{R \in R_n} \sup_{x \in (-\infty, \infty)} |\varphi(x) - R(x)|.$$

The estimate (37) generalizes the well-known result of D. Newman [10]: $R_n(|x|, [-1, 1])_C = O(e^{-\sqrt{n}})$ to the interval $(-\infty, \infty)$.

References

1. V. A. Popov, On the connection between rational and spline approximation, *C. R. Acad. Bulgare Sci.* **27** (1974), 623–626.
2. P. P. Petrushev, Connections between the rational and spline approximations in L_p metric, *Pliska Stud. Math. Bulgar.* **5** (1983), 68–83. [Russian]
3. V. A. Popov, On the connection between rational uniform and polynomial L_p approximation of functions, *in* "Quantative Approximation" (Proceedings of a Symposium on Quantitative Approximation, held in Bonn) (R. DeVore and K. Scherer, Eds.), Academic Press, New York/London, 1980.
4. P. P. Petrushev, On the rational approximation of functions, *C. R. Acad. Bulgare Sci.* **29** (1976), 1405–1408. [Russian]
5. P. P. Petrushev, Relations between rational and spline approximations, *Acta Math. Hungar.* **44** (1984), 61–83.
6. Ju. A. Brudnyi, Rational approximation and imbedding theorems, *Soviet Math. Dokl.* **20** (1979), 681–684. [Russian]
7. P. P. Petrushev, Relations between rational and spline approximations, *in* "Proceedings, International Conference on Constructive Theory of Functions, Varna, 1984," pp. 672–674.
8. I. P. Natanson, "Konstruktive Funktionentheorie," Akademie-Verlag, Berlin, 1955.
9. A. A. Goncar, On the rate of rational approximation of continuous functions with typical singularities, *Mat. Sb. (N.S.)* **73** (115) (1967), 630–638. [Russian]
10. D. J. Newman, Rational approximation to $|x|$, *Michigan Math. J.* **11** (1964), 11–14.